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# Grinberg's Criterion Applied to Some Non-Planar Graphs

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## Abstract

Robertson ([5]) and independently, Bondy ([1]) proved that the generalized Petersen graph  $P(n, 2)$  is non-hamiltonian if  $n \equiv 5 \pmod{6}$ , while Thomason [7] proved that it has precisely 3 hamiltonian cycles if  $n \equiv 3 \pmod{6}$ . The hamiltonian cycles in the remaining generalized Petersen graphs were enumerated by Schwenk [6]. In this note we give a short unified proof of these results using Grinberg's theorem.

A celebrated result of Grinberg (see [4]) concerning planar hamiltonian graphs states that if a planar graph  $G$  has a hamiltonian cycle  $C$  which partitions its  $f_i$  faces of degree  $i$  into  $f'_i$  (respectively  $f''_i$ ) faces of degree  $i$  in the interior (respectively exterior) of  $C$ , then

$$\sum_{i \geq 3} (i - 2)(f'_i - f''_i) = 0.$$

If there is precisely one natural number  $i$  not congruent to  $2 \pmod{3}$  such that  $f_i > 0$ , then Grinberg's equation cannot be satisfied, and hence the graph is non-hamiltonian. But even if the equation can be satisfied, it is still possible, in special cases, to use the criterion to prove that a graph is non-hamiltonian. Thus Thomassen [8] used the criterion to describe an infinite class of cubic planar hypohamiltonian graphs (all of which have a face partition that satisfies Grinberg's equation), and also the Tutte graph can be shown to be non-hamiltonian using the Grinberg criterion, see for example, [2] p.166 and [3] Chapter 6. In this note we apply the criterion to a class of non-hamiltonian graphs, namely some generalized Petersen graphs.

Suppose  $n$  and  $k$  are two integers such that  $1 \leq k \leq n-1$  and  $n \geq 5$ . The generalized Petersen graph  $P(n, k)$  is defined to have vertex-set  $\{u_i, v_i : i = 0, 1, \dots, n-1\}$  and edge-set  $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, 1, \dots, n-1 \text{ with subscripts reduced modulo } n\}$ .

Let  $F_m$  denote the  $m$ -th Fibonacci number defined by  $F_1 = F_2 = 1$ , and  $F_m = F_{m-1} + F_{m-2}$  for  $m > 2$ . It is easy to prove, by induction on  $n$ , that the number of matchings (including the empty matching) of the path with  $n$  vertices is  $F_{n+1}$ . Hence the number of matchings of the cycle with  $n$  vertices is  $F_{n+1} + F_{n-1}$ .

**Theorem 1** *Let  $n$  be a natural number,  $n \geq 5$ . Then*

- (i) ([1], [5])  $P(n, 2)$  is non-hamiltonian if  $n \equiv 5 \pmod{6}$ ,
- (ii) ([7])  $P(n, 2)$  has precisely three hamiltonian cycles if  $n \equiv 3 \pmod{6}$ , and
- (iii) ([6]) the number of hamiltonian cycles in  $P(n, 2)$  is

$$\begin{cases} n & \text{if } n \equiv 1 \pmod{6} \\ 2(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1) & \text{if } n \equiv 0, 2 \pmod{6} \\ n + 2(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1) & \text{if } n \equiv 4 \pmod{6} \end{cases}$$

**Proof:** Assume first that  $n \equiv 3, 5 \pmod{6}$  and  $n \geq 5$ .

If the two edges  $u_0 u_1$  and  $v_0 v_2$  are deleted from  $P(n, 2)$ , the result is a planar graph whose face-degree sequence is  $(5, \dots, 5, \frac{n+5}{2}, \frac{n+7}{2})$  (see Figure 1) and hence is non-hamiltonian by Grinberg's criterion because when  $n \equiv 3 \pmod{6}$ ,  $\frac{n+5}{2}$  and  $\frac{n+7}{2}$  are  $1, 2 \pmod{3}$  respectively while when  $n \equiv 5 \pmod{6}$ ,  $\frac{n+5}{2}$  and  $\frac{n+7}{2}$  are  $2, 0 \pmod{3}$  respectively.

This means that

(\*) if  $P(n, 2)$  has a hamiltonian cycle  $C$ , then  $C$  must contain at least one of the edges  $u_0 u_1, v_0 v_2$ .

Assume now that  $C$  is a hamiltonian cycle of  $P(n, 2)$ . Since  $C$  cannot contain all the edges of the inner cycle  $v_0 v_2 v_4 \dots v_{n-2} v_0$ , we may assume that  $v_0 v_2$  is not an edge in  $C$ . But then this implies that the paths  $v_{n-2} v_0 u_0$  and  $u_2 v_2 v_4$  must be part of  $C$ .

Since  $v_0 v_2$  is not an edge in  $C$ , the observation (\*) implies that  $u_0 u_1$  is an edge in  $C$ . By symmetry,  $u_1 u_2$  is also an edge in  $C$ . This follows because there is an automorphism (the reflection fixing  $u_1, v_1$ ) of  $P(n, 2)$  which interchanges between the edges  $u_1 u_0, u_2 u_1$  and keeps  $v_0 v_2$  fixed. ("Reflection" here refers to the standard drawing of  $P(n, k)$  where the vertices  $u_0, u_1, \dots$  and also the vertices  $v_0, v_1, \dots$  form convex  $n$ -gons.) Hence  $C$  contains the

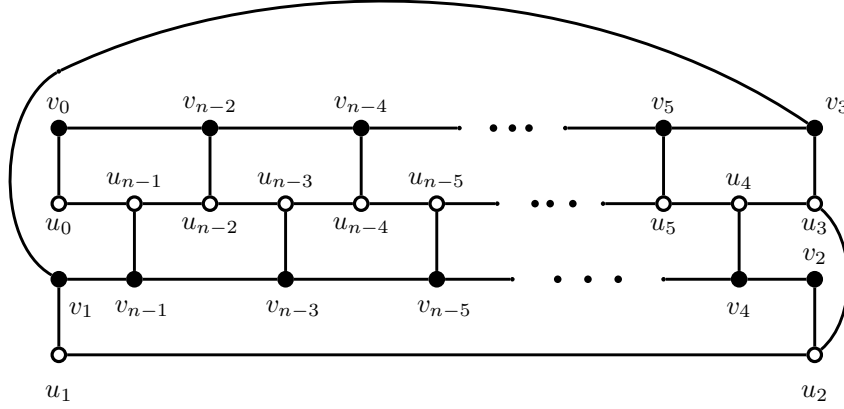


Figure 1:  $P(n, 2)$  with  $u_0u_1$  and  $v_0v_2$  deleted,  $n \equiv 3, 5 \pmod{6}$

path  $v_{n-2}v_0u_0u_1u_2v_2v_4$ . So  $C$  also contains the path  $v_{n-1}v_1v_3u_3u_4$  and therefore  $C$  does not contain the edge  $v_3v_5$ . Summarizing, we have proved that if a hamiltonian cycle  $C$  does not contain the edge  $v_0v_2$ , then  $C$  does not contain the edge  $v_3v_5$  either. By repeating this argument with  $v_3v_5$  instead of  $v_0v_2$ , we conclude that  $C$  does not contain the edge  $v_6v_8$  either. In fact  $C$  does not contain any of the edges  $v_0v_2, v_3v_5, v_6v_8, v_9v_{11}, \dots$ . Since  $C$  must contain some edge of the inner cycle  $v_0v_2v_4 \cdots v_{n-2}v_0$ , we conclude that  $n \equiv 0 \pmod{3}$ . Hence  $P(n, 2)$  has no hamiltonian cycle if  $n \equiv 5 \pmod{6}$ . In the case that  $n \equiv 3 \pmod{6}$ , the argument eventually leads to a unique hamiltonian cycle which can be rotated to yield precisely three hamiltonian cycles.

Assume next that  $n \equiv 1 \pmod{6}$ . Then Grinberg's equation is satisfied but only if the two faces of degrees  $\frac{n+5}{2}$  and  $\frac{n+7}{2}$  are both in the interior (or exterior) of the hamiltonian cycle on the resulting graph of Fig. 1. This is possible only if the edge  $v_1v_{n-1}$  is not contained in the hamiltonian cycle. Assume that  $C'$  is such a hamiltonian cycle. Then it is easy to see (from Fig. 1) that the paths  $v_{n-3}v_{n-1}u_{n-1}u_0v_0v_{n-2}u_{n-2}u_{n-3}$  and  $v_4v_2u_2u_1v_1v_3u_3u_4$  must be part of  $C'$ . There is a unique hamiltonian cycle containing these paths which can be rotated to yield  $n$  hamiltonian cycles. If it is not possible to obtain a hamiltonian cycle by deleting a pair  $u_iu_{i+1}, v_iv_{i+2}$ , then we get a contradiction as in the case when  $n \equiv 5 \pmod{6}$ .

When  $n \geq 6$  is even,  $P(n, 2)$  is a planar graph. Again, the above method can be applied. Note that  $(*)$  cannot be satisfied for each pair  $u_iu_{i+1}, v_iv_{i+2}$  and also for each pair  $u_iu_{i-1}, v_iv_{i-2}$  because the argument in the case  $n \equiv 3, 5 \pmod{6}$  would lead to a 2-factor consisting of two

cycles rather than a hamiltonian cycle. So there exists some pair of edges  $u_i u_{i+1}, v_i v_{i+2}$  (or some pair  $u_i u_{i-1}, v_i v_{i-2}$ ), say  $u_0 u_1, v_0 v_2$ , which is avoided by some hamiltonian cycle.

Draw  $P(n, 2)$  in the plane such that the  $\frac{n}{2}$ -cycle  $v_1 v_3 \dots v_{n-1} v_1$  is the outer face boundary. In this case, the face-degree sequence of the resulting graph (after deleting the edges  $u_0 u_1$  and  $v_0 v_2$ ) is  $(5, \dots, 5, \frac{n}{2}, \frac{n}{2} + 6)$  and Grinberg's equation is satisfied. For this to be possible, any hamiltonian cycle must contain the edge  $v_1 v_{n-1}$  (which is common to both the  $\frac{n}{2}$ -face and the  $(\frac{n}{2} + 6)$ -face) unless  $n \equiv 4 \pmod{6}$ .

Assume first that  $n \equiv 0, 2 \pmod{6}$ . Then for any even integer  $0 \leq i \leq \frac{n}{2}$ , whenever the pair of edges  $v_i v_{i+2}, u_i u_{i+1}$  is deleted, then the paths  $u_{i-1} u_i v_i v_{i-2}, v_{i-1} v_{i+1} u_{i+1} u_{i+2} v_{i+2} v_{i+4}$  and  $u_{i+4} u_{i+3} v_{i+3} v_{i+5}$  must be part of the hamiltonian cycle. We now follow the hamiltonian cycle along the path  $v_{i+2} v_{i+4} \dots$ . If we never use an edge  $v_j u_j$  there is a unique way to continue the hamiltonian cycle. On the other hand, the first time we use an edge  $v_j u_j$  we repeat the previous configuration with  $j$  instead of  $i$ . Those edges of the cycle  $v_0 v_2 \dots v_0$  which are not in the hamiltonian cycle clearly form a matching. On the other hand, whenever we specify a matching on this cycle, there is a unique hamiltonian cycle which avoids this matching and also avoids edges of the form  $u_i u_{i+1}, v_i v_{i+2}$ . Therefore the number of hamiltonian cycles in  $P(n, 2)$  avoiding some pairs of edges of the form  $u_i u_{i+1}, v_i v_{i+2}$  is  $F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1$ . Note that for any such hamiltonian cycle, (\*) fails for the pairs  $u_i u_{i+1}, v_i v_{i+2}$  but holds for the pairs  $u_i u_{i-1}, v_i v_{i-2}$ .

By symmetry (more precisely, by taking the cycle  $u_1 u_2 \dots u_n u_1$  in its reverse order) the number of hamiltonian cycles in  $P(n, 2)$  avoiding pairs of edges of the form  $u_i u_{i-1}, v_i v_{i-2}$  is also  $F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1$ . Thus the total number of hamiltonian cycles in  $P(n, 2)$  is  $2(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1)$  in the case  $n \equiv 0, 2 \pmod{6}$ .

The reason that we have counted all hamiltonian cycles is that (\*) fails either for the pairs  $u_i u_{i+1}, v_i v_{i+2}$  or for the pairs  $u_i u_{i-1}, v_i v_{i-2}$ , as noted above.

For  $n \equiv 4 \pmod{6}$ , there are two types of hamiltonian cycles, namely the  $2(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1)$  hamiltonian cycles which we have already counted, and also the unique hamiltonian cycle that avoids  $u_0 u_1, v_0 v_2, v_1 v_{n-1}$ , and those which can be obtained from this hamiltonian cycle by rotation and reflection. There are  $n$  hamiltonian cycles of the latter type. This completes the proof.  $\square$

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